MAC 2312 — Calculus II

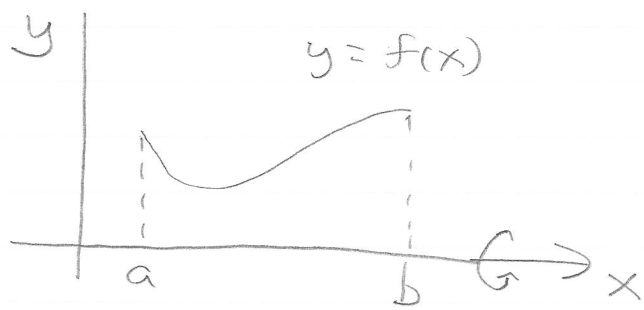
Guided Notes

Section 8.2

Surface Areas

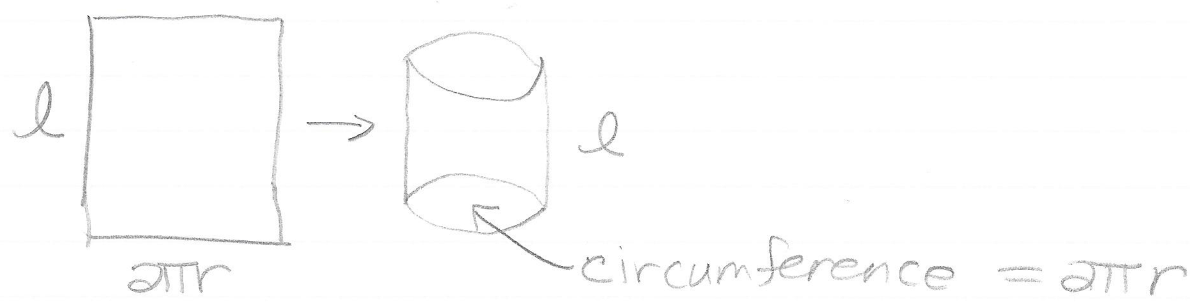
Areas of a surface of Revolution

Suppose f is a smooth, non-negative function on $[a,b]$ and that a surface of revolution is generated by revolving that portion of the curve $y=f(x)$ between $x=a$ and $x=b$ about the x-axis.



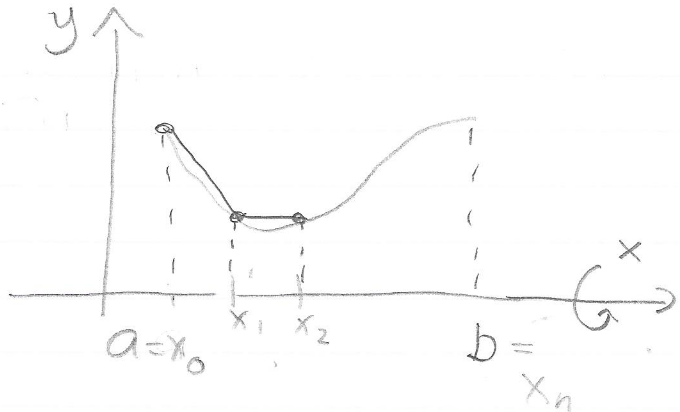
We need to define what is meant by the area S of the surface and determine a formula for it.

Let's recall a cylinder. It was created by connecting the 2 sides of a rectangle:



Now we need to extend that concept to other shapes.

consider this



Similar to shell method, but are combining exterior shells

so $S = 2\pi r l$

r: $f(x)$

l: arc length ds (see extra material for 8.1)

General Definition | Rotation about the x-axis between $x=a$, $x=b$

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx}$$

f is a smooth, non-negative function on $[a,b]$

This definition can be customized into a more friendly format and also adjusted for rotation about the y-axis radius $= y$

Rotation about the x-axis

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$S = \int_a^b 2\pi y \boxed{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

this part can be adjusted for ease of integration.

important to note
the radius must be y
$y = \ldots x \quad (y = f(x))$

Rotation about the y-axis

$$S = \int_c^d 2\pi g(y)\sqrt{1+(f'(x))^2}\,dy$$

$$S = \int_c^d 2\pi x \boxed{\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$

this part can be adjusted for ease of integration

important to note
the radius must be x
$x = \ldots y \quad (x = g(y))$

Example 1 Find the area of the surface that is generated by revolving the portion of the curve $y=x^3$ between $x=0$ and $x=1$ about the x-axis.

equation

$$\boxed{S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

entire integral must be written in terms of x

$y=x^3$

$\frac{dy}{dx}=3x^2$

$$S=\int_0^1 2\pi(x^3)\sqrt{1+(3x^2)^2}\,dx$$

$$S=\int_0^1 2\pi x^3\sqrt{1+9x^4}\,dx$$

$u=1+9x^4$

$du=36x^3dx$

$$S=\frac{2\pi}{36}\int_0^1(1+9x^4)^{1/2}(36x^3)dx = \frac{\cancel{2}\pi}{\cancel{36}_{18}}(1+9x^4)^{3/2}\cdot\frac{\cancel{2}}{3}\Big|_0^1$$

$$=\frac{\pi}{27}\left[(1+9(1)^4)^{3/2}-(1+9(0)^4)^{3/2}\right]$$

$$=\frac{\pi}{27}\left[10^{3/2}-1^{3/2}\right] = \boxed{\frac{\pi}{27}(10^{3/2}-1)}$$

$$\boxed{\cong 3.56}$$

$\boxed{\text{Example 1}}$ – Alternate approach

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

entire integral must be written in terms of y

$y = x^3$ $\qquad \frac{dx}{dy} = \frac{1}{3}y^{-2/3}$

$x = \sqrt[3]{y} = y^{1/3}$

when $x = 0$ $\quad (0,\underline{0})$ $\qquad$ when $x = 1$ $\quad (1,1)$

$y = (0)^3 = 0$ $\qquad\qquad$ $y = (1)^3 = 1$

$$S = 2\pi \int_0^1 y \sqrt{1 + \left(\tfrac{1}{3}y^{-2/3}\right)^2}\, dy = 2\pi \int_0^1 y \sqrt{1 + \frac{1}{9y^{4/3}}}\, dy$$

$$S = 2\pi \int_0^1 y \sqrt{\frac{9y^{4/3} + 1}{9y^{4/3}}}\, dy = 2\pi \int_0^1 \frac{y}{3y^{2/3}} \left(9y^{4/3} + 1\right)^{1/2} dy$$

$$S = \frac{2\pi}{3} \int_0^1 y^{1/3} \left(9y^{4/3} + 1\right)^{1/2} dy$$

$u = 9y^{4/3} + 1$

$du = \frac{9}{1} \cdot \frac{4}{3} y^{1/3}\, dy$

$du = 12y^{1/3}\, dy$

$\frac{du}{12} = y^{1/3}\, dy$

$$S = \frac{2\pi}{3 \cdot 12} \int_0^1 \left(9y^{4/3} + 1\right)^{1/2} 12y^{1/3}\, dy$$

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$$S = \frac{\pi}{18}\int_0^1 (9y^{4/3}+1)^{1/2}(12y^{1/3})\,dy$$

$$S = \frac{\pi}{18}\left[\frac{2}{3}(9y^{4/3}+1)^{3/2}\right]_0^1$$

$$S = \frac{\pi}{27}\left[(9(1)^{4/3}+1)^{3/2} - (9(0)^{4/3}+1)^{3/2}\right]$$

$$S = \frac{\pi}{27}\left[(9+1)^{3/2} - (1)^{3/2}\right]$$

$$\boxed{S = \frac{\pi}{27}(10^{3/2}-1) \simeq 3.56}$$

Example 2 Find the area of the surface that is generated by revolving the portion of the curve $y=x^2$ between $x=1$ and $x=2$ about the y-axis.

option 1

$$\boxed{S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$

from $(1,1)$ to $(2,4)$

$x=1$
$y=(1)^2=1$
$(1,1)$

when $x=2$
$y=(2)^2=4$
$(2,4)$

so since $y=x^2$ $\quad \frac{dx}{dy}=\frac{1}{2}y^{-1/2}$
$x=\sqrt{y}$

$$\boxed{S=\int_1^4 2\pi(y^{1/2})\sqrt{1+\left(\frac{1}{2y^{1/2}}\right)^2}\,dy}$$

$$S = 2\pi\int_1^4 y^{1/2}\sqrt{1+\frac{1}{4y}}\,dy = 2\pi\int_1^4 y^{1/2}\sqrt{\frac{4y+1}{4y}}\,dy$$

$$= 2\pi\int_1^4 \frac{y^{1/2}}{2y^{1/2}}\sqrt{4y+1}\,dy = \pi\int_1^4 (4y+1)^{1/2}\,dy$$

$$= \frac{2\pi}{2}(4y+1)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{4}\Big|_1^4 = \frac{\pi}{6}\left[(4y+1)^{3/2}\Big|_1^4\right]$$

$$= \frac{\pi}{6}\left[(4(4)+1)^{3/2}-(4(1)+1)^{3/2}\right] = \boxed{\frac{\pi}{6}\left[17^{3/2}-5^{3/2}\right]}$$

still about y-axis

Example 2 — option 2

$$\boxed{S = 2\pi \int_c^d x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

$y = x^2$ $\quad \frac{dy}{dx} = 2x \quad (1,1)$ to $(2,4)$

$x = \sqrt{y}$

$$S = 2\pi \int_1^2 x\left(1+(2x)^2\right)^{1/2} dx$$

u sub

$$S = 2\pi \int_1^2 x(1+4x^2)^{1/2} dx = \frac{2\pi}{8}\int_1^2 (1+4x^2)^{1/2}(8x\,dx)$$

$$S = \frac{\pi}{4}\int_1^2 (1+4x^2)^{1/2}(8x\,dx) = \frac{\pi}{4}(1+4x^2)^{3/2}\cdot\frac{2}{3}\Big|_1^2$$

$$S = \frac{\pi}{6}\left[(1+4x^2)^{3/2}\Big|_1^2\right] = \frac{\pi}{6}\left[(1+4(2)^2)^{3/2}-(1+4(1)^2)^{3/2}\right]$$

$$= \boxed{\frac{\pi}{6}\left[17^{3/2}-5^{3/2}\right]}$$

Example 3

Find the area of the surface that is generated by revolving $y = \frac{x^3}{6} + \frac{1}{2x}$ about the x-axis on the interval $[1, 2]$

starting x

ending x

for revolving about x-axis we start with one of the following

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

option 1

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

option 2

$$y = \frac{x^3}{6} + \frac{1}{2x} = \frac{2x^4 + 6}{12x} = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$$

lcd $= 12x$

$$\boxed{y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}}$$

$$\frac{dy}{dx} = \frac{1}{6}(3)x^2 + \frac{1}{2}(-1)x^{-2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}}$$

alternative

$$y = \frac{2x^4 + 6}{12x}$$

$$12xy = 2x^4 + 6$$

$$12xy - 2x^4 = 6$$

we need to solve for x which is not user friendly, so $\boxed{\text{option 1}}$ is our best choice based on this data.

$x = 1 \quad y = \frac{1}{6} + \frac{1}{2} = \frac{2}{12} + \frac{6}{12} = \frac{8}{12} = \frac{3}{4}$

$x = 2 \quad y = \frac{1}{6}(2)^3 - \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{8}{6} - \frac{1}{4} = \frac{16}{12} - \frac{3}{12} = \frac{13}{12}$

$$\boxed{(1, 3/4) \text{ to } (2, \tfrac{13}{12})}$$

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$$S = 2\pi \int_1^2 y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\left(\frac{x^4}{4}-2\left(\frac{\cancel{x^2}}{2}\right)\left(\frac{1}{2\cancel{x^2}}\right)+\frac{1}{4x^4}\right)}\,dx$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\frac{x^4}{4}-\frac{1}{2}+\frac{1}{4x^4}}\,dx$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{\frac{1}{2}+\frac{x^4}{4}+\frac{1}{4x^4}}\,dx \qquad lcd = 4x^4$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}-\frac{1}{2x}\right)\sqrt{\frac{2x^4+x^8+1}{4x^4}}\,dx$$

numerator = perfect square

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}-\frac{1}{2x}\right)\sqrt{\frac{(x^4+1)^2}{4x^4}}\,dx$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}-\frac{1}{2x}\right)\frac{(x^4+1)}{2x^2}\,dx$$

multiply and finish

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6}-\frac{1}{2x}\right)\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)dx = \boxed{\frac{47\pi}{16}}$$

Section 8.2 Areas of a Surface of Revolution	Practice Problems

Find the area of the surfaces generated given specific conditions.

① $y = \frac{1}{3}x^3$ on the interval $[0, 3]$ about the x-axis

② $y = \sqrt[3]{x} + 2$ on the interval $[1, 8]$ about the y-axis

③ $y = \sin x$ on the interval $[0, \pi]$ about the x-axis

use the calculator to integrate this one

④ $y = e^{-x}$ from $x = 0$ to $x = 1$ about the x-axis

⑤ $y = \ln x$ from $x = 1$ to $x = 2$ about the y-axis

Section 8.2 Areas of surface of Revolution	practice problem answers and helpful hints

① $S = 2\pi \int_0^3 \frac{1}{3}x^3\sqrt{1+x^4}\,dx = \frac{\pi}{9}\left(82\sqrt{82}-1\right)$

$\cong 258.85$

② $S = 2\pi \int_1^8 x\sqrt{1+\frac{1}{9x^{4/3}}}\,dx = \frac{\pi}{27}\left(145\sqrt{145}-10\sqrt{10}\right)$

$\cong 199.48$

③ 14.424

④ $S = 2\pi \int_0^1 e^{-x}\sqrt{1+(e^{-x})^2}\,dx = 2\pi \int_0^1 e^{-x}\sqrt{1+e^{-2x}}\,dx$

hint: use trig substitution

final answer $\pi\left[\sqrt{2}+\ln(1+\sqrt{2}) - e^{-1}\sqrt{e^{-2}+1} - \ln\left(e^{-1}+\sqrt{e^{-2}+1}\right)\right]$

⑤ also use trig substitution

$S = 2\pi \int_1^2 \sqrt{x^2+1}\,dx$

square both sides (s^2) to make simplification easier

$\boxed{\text{final answer}}$

$\pi\left(\left[2\sqrt{5}+\ln|2+\sqrt{5}|\right] - \left[\sqrt{2}+\ln|1+\sqrt{2}|\right]\right)$

Further Exploration

parametric equations approach

Let's make some more connections to the surface area $\underline{ds}$

Alternate version

x-axis

Let a smooth curve C be given by $x=f(t)$ and $y=g(t)$ [parametric equations] where $a \le t \le b$, and suppos that $g(t) \ge 0$ for all t. The area S of the surface of revolution obtained by revolving curve C about the x-axis is

$$S = \int_a^b 2\pi g(t)\sqrt{(f'(t))^2 + (g'(t))^2}\,dt = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

y

from our 8.1 explorations, we know that

$$(ds)^2 = (dx)^2 + (dy)^2 \qquad \text{where } ds = \text{arc length}$$

so $ds = \sqrt{(dx)^2 + (dy)^2}$ but $dx \to \frac{dx}{dt}$

$dy \to \frac{dy}{dt}$

y-axis

$$S = \int_a^b 2\pi f(t)\sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$\boxed{\text{Take a look at this}}$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ has some interesting results when we return it to the rectangular coordinate system

Recall that

$$\boxed{\frac{dy}{dx}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dy}{dx}\left(\frac{dx}{dt}\right) = \frac{dy}{dt} \qquad \frac{dx}{dt}\left(\frac{dy}{dx}\right) = \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{dx}{dy}$$

$\boxed{\text{Let's rewrite without paramater } t}$

$\boxed{\text{substitute } \frac{dx}{dt} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dy}{dx}\right)}}$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dy}{dx}\right)^2} + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= \sqrt{\left(\frac{dy}{dt}\right)^2\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= \sqrt{\left(\frac{dy}{dt}\right)^2\left[\left(\frac{dx}{dy}\right)^2 + 1\right]}\; dt$$

$$= \sqrt{\left(\frac{dy}{dt}\right)^2}\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dt$$

$$= \frac{dy}{\cancel{dt}}\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; \cancel{dt}$$

$$\boxed{\text{so } \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

If you substitute $\frac{dy}{dt} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right)$ into $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

you get:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2}\,dt$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2\left(\frac{dx}{dt}\right)^2}\,dt$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\,dt$$

$$= \frac{dx}{\cancel{dt}}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\cancel{dt} = \boxed{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

So, if we look at arc length (and hence area of a surface rotated about an axis) from a parametric equation point of view, you can see why we have the option to make adjustments to our area equations when we are using the cartesean coordinate system.